Ring and Module Structures on the K-Theory of C^*-Algebras from Smale Spaces

Brady Killough

Mount Royal University

Smale Space, their groupoids, and C^*-algebras
May 15, 2022

Overview

- Smale space Preliminaries
- K-theory of a commutative C^*-algebra
- The mapping Cylinder
- Ring and Module Structures
- The SFT case
- Duality for SFT

Smale space and C*-algebras

- (X, φ) a mixing Smale space
- P a finite φ-invariant subset of X
- groupoids $G^s(X, \varphi, P)$, $G^u(X, \varphi, P)$, $G^h(X, \varphi)$,
- $S(X, \varphi, P)$, $U(X, \varphi, P)$, $H(X, \varphi)$ the associated C^*-algebras
- we consider each of these as subalgebras of $\mathcal{B}(l^2(V^h(P)))$

$*$-automorphism

For $f \in C_c(G)$ where G is any one of the three groupoids from the previous slide, we define

$$\alpha(f)(x, y) = f(\varphi^{-1}(x), \varphi^{-1}(y)).$$

α extends to a $*$-automorphism on each of $S(X, \varphi, P)$, $U(X, \varphi, P)$, and $H(X, \varphi)$.

Assymptotic Commutation Relations

Let $a \in S(X, \varphi, P)$, $b \in U(X, \varphi, P)$, and $f, g \in H(X, \varphi)$. Then

- $af, fa \in S(X, \varphi, P)$,
- $bf, fb \in U(X, \varphi, P)$,
- $\lim_{n \to +\infty} ||a\alpha^{-n}(f) - \alpha^{-n}(f)a|| = 0$
- $\lim_{n \to +\infty} ||\alpha^{n}(f)b - b\alpha^{n}(f)|| = 0$
- $\lim_{n \to +\infty} ||\alpha^{n}(f)\alpha^{-n}(g) - \alpha^{-n}(g)\alpha^{n}(f)|| = 0$

K-Theory of a commutative C*-Algebra

- Suppose A is a commutative C*-Algebra, p, q projections in A. Then $pq = qp$ is a projection in A.
- For $p \in \mathbb{M}_n(A)$, $p \in \mathbb{M}_m(A)$, we define $p \times q \in M_{nm}(A)$ entrywise by $(p \times q)_{(i,j)(k,l)} = p_{ik}q_{kl}$. Then $[p]_0[q]_0 = [p \times q]_0$ defines a ring structure on $K_0(A)$
- Noting that $C(S^1, A)$ is also commutative, and that $K_0(C(S^1, A)) \cong K_0(A) \oplus K_1(A)$, we can use the above idea to define a product on $K_*(A)$.

K-Theory of a commutative C*-Algebra

- Suppose A is a commutative C*-Algebra, p, q projections in A. Then $pq = qp$ is a projection in A.
- For $p \in \mathbb{M}_n(A)$, $p \in \mathbb{M}_m(A)$, we define $p \times q \in M_{nm}(A)$ entrywise by $(p \times q)_{(i,j)(k,l)} = p_{ik}q_{kl}$. Then $[p]_0[q]_0 = [p \times q]_0$ defines a ring structure on $K_0(A)$
- Noting that $C(S^1, A)$ is also commutative, and that $K_0(C(S^1, A)) \cong K_0(A) \oplus K_1(A)$, we can use the above idea to define a product on $K_*(A)$.
- $K_*(A)$ is a commutative ring.

Back to Smale Spaces

The C^*-Algebras defined previously are not commutative, but the asymptotic commutation relations are encouraging.

For $p, q \in P_1(H(X, \varphi))$, n large, we define

$$a_n = \frac{\alpha^n(p)\alpha^{-n}(q) + \alpha^{-n}(q)\alpha^n(p)}{2}.$$

Then $a_n = a_n^*$, $||a_n^2 - a_n||$ is small, and $\chi_{(1/2,\infty)}(a_n)$ is a projection.

Question: Can we define $[p]_0[q]_0 = \lim_{n\to\infty}[a_n]_0$?

Back to Smale Spaces

The C^*-Algebras defined previously are not commutative, but the asymptotic commutation relations are encouraging.

For $p, q \in P_1(H(X, \varphi))$, n large, we define

$$a_n = \frac{\alpha^n(p)\alpha^{-n}(q) + \alpha^{-n}(q)\alpha^n(p)}{2}.$$

Then $a_n = a_n^*$, $||a_n^2 - a_n||$ is small, and $\chi_{(1/2,\infty)}(a_n)$ is a projection.

Question: Can we define $[p]_0[q]_0 = \lim_{n\to\infty}[a_n]_0$?

No, a_n typically depends on n, even for arbitrarily large n, so $\lim_{n\to\infty}[a_n]_0$ is not well defined.

The Mapping Cylinder

Consider the mapping cylinder,

- $C(H, \alpha) = \{f : \mathbb{R} \to H(X, \varphi) \mid f(t+1) = \alpha(f(t)) \ \forall t\}$, and
- $\alpha_t(f)(s) = f(s+t)$ so that $\alpha_n(f)(s) = \alpha^n(f(s))$ for $n \in \mathbb{Z}$.

The Mapping Cylinder

Consider the mapping cylinder,

- $C(H, \alpha) = \{f : \mathbb{R} \to H(X, \varphi) \mid f(t+1) = \alpha(f(t)) \ \forall t\}$, and
- $\alpha_t(f)(s) = f(s+t)$ so that $\alpha_n(f)(s) = \alpha^n(f(s))$ for $n \in \mathbb{Z}$.

For $f, g \in C(H, \alpha)$ let

$$(f \times g)_t = \frac{\alpha_t(f)\alpha_{-t}(g) + \alpha_{-t}(g)\alpha_t(f)}{2}.$$

The Mapping Cylinder

Consider the mapping cylinder,

- $C(H,\alpha) = \{f : \mathbb{R} \to H(X,\varphi) \mid f(t+1) = \alpha(f(t))\ \forall t\}$, and
- $\alpha_t(f)(s) = f(s+t)$ so that $\alpha_n(f)(s) = \alpha^n(f(s))$ for $n \in \mathbb{Z}$.

For $f, g \in C(H,\alpha)$ let

$$(f \times g)_t = \frac{\alpha_t(f)\alpha_{-t}(g) + \alpha_{-t}(g)\alpha_t(f)}{2}.$$

For $f \in M_n(C(H,\alpha))$, $g \in M_m(C(H,\alpha))$, $(f \times g)_t \in M_{nm}(C(H,\alpha))$ is

$$((f \times g)_t)_{(ij)(i'j')} = (f_{ii'} \times g_{jj'})_t$$

For $p \in P_n(C(H,\alpha))$, $q \in P_m(C(H,\alpha))$, there exists T such that for $t \geq T$

$$\chi_{(1/2,\infty)}(p \times q)_t \in P_{nm}(C(H,\alpha)).$$

So we get a continuous path of projections and

$$[p]_0[q]_0 = \lim_{t\to\infty}[(p \times q)_t]_0$$

is well defined and gives a product on $K_0(C(H,\alpha))$.

For $p \in P_n(C(H,\alpha))$, $q \in P_m(C(H,\alpha))$, there exists T such that for $t \geq T$

$$\chi_{(1/2,\infty)}(p \times q)_t \in P_{nm}(C(H,\alpha)).$$

So we get a continuous path of projections and

$$[p]_0[q]_0 = \lim_{t\to\infty}[(p \times q)_t]_0$$

is well defined and gives a product on $K_0(C(H,\alpha))$.

To define a product on $K_*(C(H,\alpha))$ we note that the asymptotically abelian property is inherited by $C(S^1, C(H,\alpha))$ and consider the split exact sequence

$$0 \longrightarrow S(C(H,\alpha)) \hookrightarrow C(S^1, C(H,\alpha)) \longrightarrow C(H,\alpha) \longrightarrow 0$$

so $K_0(C(S^1, C(H,\alpha))) \cong K_0(C(H,\alpha)) \oplus K_1(C(H,\alpha))$ and defining a product on $K_0(C(S^1, C(H,\alpha)))$ (as on the previous slide), then gives a product structure on $K_*(C(H,\alpha))$.

$K_*(C(H,\alpha))$

To define a product on $K_*(C(H,\alpha))$ we note that the asymptotically abelian property is inherited by $C(S^1, C(H,\alpha))$ and consider the split exact sequence

$$0 \longrightarrow S(C(H,\alpha)) \hookrightarrow C(S^1, C(H,\alpha)) \longrightarrow C(H,\alpha) \longrightarrow 0$$

so $K_0(C(S^1, C(H,\alpha))) \cong K_0(C(H,\alpha)) \oplus K_1(C(H,\alpha))$ and defining a product on $K_0(C(S^1, C(H,\alpha)))$ (as on the previous slide), then gives a product structure on $K_*(C(H,\alpha))$.

- $K_*(C(H,\alpha))$ is a (potentially non-commutative) ring.

How do we compute $K_*(C(H,\alpha))$?

$$0 \longrightarrow S(H(X,\varphi)) \xrightarrow{\iota} C(H,\alpha) \xrightarrow{e_0} H(X,\varphi) \to 0$$

- $\iota(f)(s) = \alpha^k(f(s-k))$
- e_0 : evaluation at 0.

How do we compute $K_*(C(H,\alpha))$?

$$0 \longrightarrow S(H(X,\varphi)) \xrightarrow{\iota} C(H,\alpha) \xrightarrow{e_0} H(X,\varphi) \to 0$$

- $\iota(f)(s) = \alpha^k(f(s-k))$
- e_0 : evaluation at 0.

$$K_1(H(X,\varphi)) \cong K_0(S(H(X,\varphi))) \longrightarrow K_0(C(H,\alpha)) \longrightarrow K_0(H(X,\varphi))$$

$$K_0(H(X,\varphi)) \xrightarrow{id-\alpha_*} K_0(H(X,\varphi)) \cong K_1(S(H(X,\varphi)))$$

$$K_1(H(X,\varphi)) \longleftarrow K_1(C(H,\alpha)) \longleftarrow K_1(S(H(X,\varphi))) \cong K_0(H(X,\varphi))$$

$$K_1(H(X,\varphi)) \xrightarrow{id-\alpha_*} K_1(H(X,\varphi)) \cong K_0(S(H(X,\varphi)))$$

How do we compute $K_*(C(H,\alpha))$?

$$0 \longrightarrow S(H(X,\varphi)) \xrightarrow{\iota} C(H,\alpha) \xrightarrow{e_0} H(X,\varphi) \to 0$$

- $\iota(f)(s) = \alpha^k(f(s-k))$
- e_0 : evaluation at 0.

$$\begin{array}{ccccccc}
K_1(H(X,\varphi)) & \cong K_0(S(H(X,\varphi))) & \longrightarrow & K_0(C(H,\alpha)) & \longrightarrow & K_0(H(X,\varphi)) & \\
\uparrow{\scriptstyle id-\alpha_*} & & & & & \downarrow{\scriptstyle id-\alpha_*} & \\
K_1(H(X,\varphi)) & \longleftarrow & K_1(C(H,\alpha)) & \longleftarrow & K_1(S(H(X,\varphi))) \cong & K_0(H(X,\varphi)) &
\end{array}$$

In general, this is difficult.

Shift of Finite type

If $(X, \varphi) = (\Sigma, \sigma)$ is a SFT, then $H(\Sigma, \sigma)$ is AF, so $K_1(H(\Sigma, \sigma)) = 0$ and we have

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(H,\alpha)) & \longrightarrow & K_0(H(\Sigma,\sigma)) \\ & & & & \Big\downarrow id - \alpha_* \\ 0 & \longleftarrow & K_1(C(H,\alpha)) & \longleftarrow & K_0(H(\Sigma,\sigma)) \end{array}$$

Shift of Finite type

If $(X, \varphi) = (\Sigma, \sigma)$ is a SFT, then $H(\Sigma, \sigma)$ is AF, so $K_1(H(\Sigma, \sigma)) = 0$ and we have

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(C(H,\alpha)) & \longrightarrow & K_0(H(\Sigma,\sigma)) \\ & & & & \Big\downarrow id - \alpha_* \\ 0 & \longleftarrow & K_1(C(H,\alpha)) & \longleftarrow & K_0(H(\Sigma,\sigma)) \end{array}$$

- $K_0(C(H, \alpha)) \cong ker(id - \alpha_*)$
- $K_1(C(H, \alpha)) \cong K_0(H(\Sigma, \sigma))/image(id - \alpha_*)$

Inductive Limits

Suppose (Σ_A, σ) is the edge shift with graph $G = (V, E)$ and adjacency matrix A, then $K_0(H(\Sigma_A, \sigma))$ is the inductive limit of

$$M_{\# V}(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_{\# V}(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_{\# V}(\mathbb{Z}) \longrightarrow \cdots$$

Inductive Limits

Suppose (Σ_A, σ) is the edge shift with graph $G = (V, E)$ and adjacency matrix A, then $K_0(H(\Sigma_A, \sigma))$ is the inductive limit of

$$M_{\#V}(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_{\#V}(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_{\#V}(\mathbb{Z}) \longrightarrow \cdots$$

$$K_0(H(\Sigma_A, \sigma)) \cong (M_{\#V}(\mathbb{Z}) \times \mathbb{N}) / \sim$$

- For $n \leq m$, $(X, n) \sim (Y, m)$ if and only if $A^{m-n+k} X A^{m-n+k} = A^k Y A^k$ for some k.
- We denote the equivalence class $[X, n]$.

Inductive Limits

Suppose (Σ_A, σ) is the edge shift with graph $G = (V, E)$ and adjacency matrix A, then $K_0(H(\Sigma_A, \sigma))$ is the inductive limit of

$$M_{\#V}(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_{\#V}(\mathbb{Z}) \xrightarrow{X \mapsto AXA} M_{\#V}(\mathbb{Z}) \longrightarrow \cdots$$

$$K_0(H(\Sigma_A, \sigma)) \cong (M_{\#V}(\mathbb{Z}) \times \mathbb{N}) / \sim$$

- For $n \leq m$, $(X, n) \sim (Y, m)$ if and only if $A^{m-n+k} X A^{m-n+k} = A^k Y A^k$ for some k.
- We denote the equivalence class $[X, n]$.
- Then $\alpha([X, n]) = [XA^2, n+1]$, $\alpha^{-1}([X, n]) = [A^2 X, n+1]$

Inductive Limits

Let $C(A) = \{X \in M_{\#V}(\mathbb{Z}) \mid AX = XA\}$, then

$$K_0(C(H,\alpha)) \cong ker(id - \alpha_*) \cong \varinjlim C(A)$$

Inductive Limits

Let $C(A) = \{X \in M_{\#V}(\mathbb{Z}) \mid AX = XA\}$, then

$$K_0(C(H,\alpha)) \cong ker(id - \alpha_*) \cong \lim_{\to} C(A)$$

Let $B(A) = \{X \mid X = YA - AY, \text{ for some } Y \in M_{\#V}(\mathbb{Z})\}$, then

$$\lim_{\to} B(A) \cong image(id - \alpha_*) \subset K_0(H(\Sigma_A, \sigma))$$

Let $C(A) = \{X \in M_{\#V}(\mathbb{Z}) \mid AX = XA\}$, then

$$K_0(C(H,\alpha)) \cong ker(id - \alpha_*) \cong \lim_{\rightarrow} C(A)$$

Let $B(A) = \{X \mid X = YA - AY, \text{ for some } Y \in M_{\#V}(\mathbb{Z})\}$, then

$$\lim_{\rightarrow} B(A) \cong image(id - \alpha_*) \subset K_0(H(\Sigma_A, \sigma))$$

so

$$K_1(C(H,\alpha)) \cong K_0(H(\Sigma_A, \sigma)) / \lim_{\rightarrow} B(A) \cong \lim_{\rightarrow}(M_{\#V}(\mathbb{Z})/B(A))$$

Product formula

- $K_0(C(H, \alpha)) : [X, N]$ s.t. $X \in C(A)$
- $K_1(C(H, \alpha)) : [Y + B(A), M]$

Product formula

- $K_0(C(H,\alpha))$: $[X,N]$ s.t. $X \in C(A)$
- $K_1(C(H,\alpha))$: $[Y+B(A),M]$

If $[X_1,N],[Y_1,M] \in K_0(C(H,\alpha)$, $[X_2+B(A),N],[Y_2+B(A),M] \in K_1(C(H,\alpha)$, then

- $[X_1,N] * [Y_1,M] = [X_1Y_1,N+M]$
- $[X_1,N] * [Y_2+B(A),M] = [X_1Y_2+B(A),N+M]$
- $[X_2+B(A),N] * [Y_1,M] = [X_2Y_1+B(A),N+M]$
- $[X_2+B(A),N] * [Y_2+B(A),M] = 0$

Product formula

- $K_0(C(H,\alpha)) : [X, N]$ s.t. $X \in C(A)$
- $K_1(C(H,\alpha)) : [Y + B(A), M]$

If $[X_1, N], [Y_1, M] \in K_0(C(H,\alpha)$, $[X_2 + B(A), N], [Y_2 + B(A), M] \in K_1(C(H,\alpha)$, then

- $[X_1, N] * [Y_1, M] = [X_1 Y_1, N + M]$
- $[X_1, N] * [Y_2 + B(A), M] = [X_1 Y_2 + B(A), N + M]$
- $[X_2 + B(A), N] * [Y_1, M] = [X_2 Y_1 + B(A), N + M]$
- $[X_2 + B(A), N] * [Y_2 + B(A), M] = 0$

$$
\begin{aligned}
([X_1, N] + [X_2 + B(A), N]) * ([Y_1, M] + [Y_2 + B(A), M]) \\
= [X_1 Y_1, N + M] + [X_1 Y_2 + X_2 Y_1 + B(A), N + M]
\end{aligned}
$$

Example

$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ Then $C(A)$ is spanned by

$$X_1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, X_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$X_5 = I$

$$0 \longrightarrow \mathbb{Z}^4 \longrightarrow K_0(C(H, \alpha)) \longrightarrow \mathbb{Z}[1/2] \longrightarrow 0$$

$< X_1, X_2, X_3, X_4 > \cong \mathbb{Z}^4$ is a non-commutative ideal in $K_0(C(H, \alpha))$.

$K_0(S)$, $K_0(U)$

- $K_0(S(\Sigma_A, \sigma))$:

$$\mathbb{Z}^{\#V(G)} \xrightarrow{v \mapsto vA} \mathbb{Z}^{\#V(G)} \xrightarrow{v \mapsto vA} \mathbb{Z}^{\#V(G)} \longrightarrow \dots$$

$\alpha[v, N] = [vA, N]$, $\alpha^{-1}[v, N] = [v, N+1]$,

- $K_0(U(\Sigma_A, \sigma))$:

$$\mathbb{Z}^{\#V(G)} \xrightarrow{v \mapsto Av} \mathbb{Z}^{\#V(G)} \xrightarrow{v \mapsto Av} \mathbb{Z}^{\#V(G)} \longrightarrow \dots$$

$\alpha[w, M] = [w, M+1]$, $\alpha^{-1}[w, M] = [Aw, M]$

Module Structure

Let $[X, N] \in K_0(C(H, \alpha)$, $[v, M] \in K_0(S(\Sigma_A, \sigma))$, $[w, M] \in K_0(U(\Sigma_A, \sigma))$, then

- $[v, M] * [X, N] = [vX, M + 2N]$
- $[X, N] * [w, M] = [Xw, M + 2N]$

Dual Modules

For $[v, M] \in K_0(S(\Sigma_A, \sigma))$

- $\alpha[v, M] = [vA, M] = [v, M] * [A, 0]$
- $\alpha^{-1}[v, M] = [v, M + 1] = [vA, M + 2] = [v, M] * [A, 1]$

Dual Modules

For $[v, M] \in K_0(S(\Sigma_A, \sigma))$

- $\alpha[v, M] = [vA, M] = [v, M] * [A, 0]$
- $\alpha^{-1}[v, M] = [v, M+1] = [vA, M+2] = [v, M] * [A, 1]$

Let R be the subring generated by $[A, 0],\ [A, 1]$. Then

$$Hom_R(K_0(S(\Sigma_A, \sigma)), R) \cong K_0(U(\Sigma_A, \sigma))$$